数理解析研究所講究録 1391 巻 2004 年 21-33

Relationships between BC-s.d.$\Omega(f)$ and (K,ρ)-s.d.$\Omega(f)$ in a functional differential equation with infinite delay

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Abstract

In order to obtain the existence of an almost periodic solution to the functional differential equation $\dot{x}(t)=f(t,x_t)$, where x_t is defined by $x_t(s)=x(t+s)$ for $t\in R^-$, on a fading memory space B, we consider a certain stability property which is referred to as BC-stable under disturbances from hull. This stability implies ρ-stable under disturbances from hull with respect to compact set K.

1 Introduction

For the ordinary differential equations and functional differential equations, the existence of almost periodic solutions of almost periodic systems has been studied by many authors. One of the most popular method is to assume the certain stability properties [6,7,8,10]. Recently, [5] has shown the existence of almost periodic solutions of the abstract functional differential equations on a fading memory space by assuming the existence of a bounded solution which is BC total stable. In this paper, in order to obtain existence theorems for an almost periodic solution to the functional differential equation with infinite delay, we discusse to improve Hamaya's results [1,2] and Murakami and Yoshizawa's result [9], as a corollary, to theorems for the functional differential equations.

Let R^n denote the n-dimensional real linear space and $|\cdot|$ will denote the appropriate norm in R^n. For any interval $I\subset R:=(-\infty,\infty)$, we denote by $BC=BC(I)$ the set of all bounded continuous functions mapping I into R^n and set $|\phi|_{BC}=\sup\{|\phi(s)|: s\in I\}$ when $I=(-\infty,0]$. Now, for any function $x:(-\infty,a)\to R^n$ and $t<a$, define a function

$x_t : R^- := (-\infty, 0] \to R^n$ by $x_t(s) = x(t+s)$ for $s \in R^-$. Let B be a real linear space of functions mapping R^- into R^n with a complete seminorm $|\cdot|_B$. We assume the following conditions on the space B.

(A1) There exist positive constants J, L and M with the property that if $x : (-\infty, a) \to R^n$ is continuous on $[\sigma, a)$ with $x_\sigma \in B$ for some $\sigma < a$, then for all $t \in [\sigma, a)$,

(*i*) $x_t \in B$,

(*ii*) x_t is continuous in t (with respect to $|\cdot|_B$),

(*iii*) $J|x(t)| \le |x_t|_B \le L \sup_{\sigma \le s \le t} |x(s)| + M|x_\sigma|_B$,

(A2) If $\{\phi^k\}$ is a sequence in $B \cap BC$ converging to a function ϕ uniformly on any compact interval in R^- and $\sup_k |\phi^k|_{BC} < \infty$, then $\phi \in B$ and $|\phi^k - \phi|_B \to 0$ as $k \to \infty$.

We hold that the space B contains BC and that there is a constant $l > 0$ such that

$$|\phi|_B \le l|\phi|_{BC} \qquad \text{for all} \quad \phi \in BC. \tag{1}$$

The space B is called a fading memory space, if it satisfies the following fading memory condition together with (A1) and (A2).

(A3) If $x : R \to R^n$ is a function such that $x_0 \in B$, and $x(t) \equiv 0$ on $R^+ := [0, \infty)$,

then $|x_t|_B \to 0$ as $t \to \infty$.

It is well known ([6]) that the following typical example of fading memory spaces. Let $g : R^- \to [1, \infty)$ be any continuous nonincreasing function such that $g(0) = 1$ and $g(s) \to \infty$ as $s \to -\infty$. We set

$$C_g^0 = \{\phi : R^- \to R^n \text{ is continuous with } \lim_{s \to -\infty} |\phi(s)|/g(s) = 0\}.$$

Then the space C_g^0 equipped with the norm

$$|\phi|_g = \sup_{s \le 0} |\phi(s)|/g(s), \qquad \phi \in C_g^0,$$

is a separable Banach space and satisfies (A1),(A2) and (A3). We introduce an almost periodic function $f(t, x) : R \times B \to R^n$.

Definition 1. $f(t, x)$ is said to be almost periodic in t uniformly for $x \in B$, if for any $\epsilon > 0$ and any compact set K in B, there exists a positive number $L^*(\epsilon, K)$ such that any interval of length $L^*(\epsilon, K)$ contains a τ for which

$$|f(t+\tau, x) - f(t, x)| \le \epsilon \tag{2}$$

for all $t \in R$ and all $x \in K$. Such a number τ in (2) is called an ϵ-translation number of $f(t, x)$.

In order to formulate a property of almost periodic functions, which is equivalent to the above difinition, we discuss the concept of the normality of almost periodic functions. Namely, Let $f(t,x)$ be almost periodic in t uniformly for $x \in B$. Then, for any sequence $\{h_k'\} \subset R$, there exists a subsequence $\{h_k\}$ of $\{h_k'\}$ and function $g(t,x)$ such that

$$f(t+h_k, x) \rightarrow g(t,x) \tag{3}$$

uniformly on $R \times K$ as $k \rightarrow \infty$, where K is any compact set in B. We shall denote by $T(f)$ the function space consisting of all translates of f, that is, $f_\tau \in T(f)$, where

$$f_\tau(t,x) = f(t+\tau, x), \qquad \tau \in R \tag{4}$$

Let $H(f)$ denote the closure of $T(f)$ in the sense of (4). $H(f)$ is called the hull of f. In particular, we denote by $\Omega(f)$ the set of all limit functions $g \in H(f)$ such that for some sequence $\{t_k\}, t_k \rightarrow \infty$ as $k \rightarrow \infty$ and $f(t+t_k, x) \rightarrow g(t,x)$ uniformly on $R \times S$ for any compact subset S in B. By (3), if $f : R \times B \rightarrow R^n$ is almost periodic in t uniformly for $x \in B$, so is a function in $\Omega(f)$. The following concept of asymptotic almost periodicity was introduced by Frechet in the case of continuous function (cf.[10]).

Definition 2. Let $u : R^+ \rightarrow R^n$ be a continuous function. $u(t)$ is said to be asymptotically almost periodic if it is a sum of an almost periodic function $p(t)$ and a continuous function $q(t)$ defined on R^+ which tends to zero as $t \rightarrow \infty$, that is,

$$u(t) = p(t) + q(t).$$

$u(t)$ is asymptotically almost periodic if and only if for any sequence $\{t_k\}$ such that $t_k \rightarrow \infty$ as $k \rightarrow \infty$ there exists a subsequence $\{t_{k_j}\}$ for which $u(t+t_{k_j})$ converges uniformly on R^+.

2 Existence of almost periodic solutions

We shall consider the almost periodic functional differential equation

$$\dot{x}(t) = f(t, x_t) \qquad t \in R^+, \tag{5}$$

where $f : R^+ \times B \rightarrow R^n$. We impose the following assumptions on (5):

(H1) For any $H > 0$, there is an $L_0(H) > 0$ such that $\sup|f(t,\phi)| \leq L_0(H)$ for all $t \in R^+$ and $|\phi|_B \leq H$.

(H2) $f(t,\phi)$ is uniformly continuous in $(t,\phi) \in R^+ \times K$ for any compact set K in B, and almost periodic in t uniformly for $x \in B$.

(H3) Eq.(5) has a bounded solution u defined on R^+ which passes through $(0, u_0)$, that is $\sup|u(t)| < \infty$ for all $t \in R^+$ and $u_0 \in BC$. We can see from (H3) and (A1) that $\sup_{t\geq 0}|u_t|_B < \infty$ and hence $\sup_{t\geq 0}|\dot{u}(t)| < \infty$ by (H1). Thus the set

$$\Gamma(u) := \text{the closure of } \{u_t : t \in R^+\}$$

is compact in B (cf.[6,7,8]).

Now we introduce BC-stability properties and ρ-stability properties with respect to the compact set K and the metric ρ.

Definition 3. The bounded solution $u(t)$ of Eq.(5) is said to be BC-totally stable (in short, BC-TS) if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $t_0 \geq 0$, $|x_{t_0} - u_{t_0}|_{BC} < \delta(\epsilon)$ and $h \in BC([t_0, \infty))$ which satisfies $\sup_{t\in[t_0,\infty)}|h(t)| < \delta(\epsilon)$, then $|x(t) - u(t)| < \epsilon$ for all $t \geq t_0$, where $x(t)$ is a solution of

$$\dot{x}(t) = f(t, x_t) + h(t) \qquad t \in R^+ \tag{6}$$

through (t_0, ϕ) such that $x_{t_0}(s) = \phi(s)$ for all $s \leq 0$.

Let K be the compact set in R^n such that $u(t) \in K$ for all $t \in R$, where $u(t) = \phi^0(t)$ for $t < 0$. For any $\theta, \psi \in BC$, we set

$$\begin{aligned}\rho(\theta, \psi) &= \sum_{j=1}^{\infty} \rho_j(\theta, \psi)/[2^j(1 + \rho_j(\theta, \psi))], \\ \text{where} & \\ \rho_j(\theta, \psi) &= \sup_{-j\leq s\leq 0} |\theta(s) - \psi(s)|.\end{aligned}$$

Clearly, $\rho(\theta^k, \theta) \to 0$ as $k \to \infty$ if and only if $\theta^k(s) \to \theta(s)$ uniformly on any compact subset of $(-\infty, 0]$ as $k \to \infty$. We set $O(u) =$ the closure of $\{u(t) : t \in R\}$, and we consider any compact set K in R^n such that interior $K \supset O(u)$.

Definition 4. The bounded solution $u(t)$ of Eq.(5) is said to be (K, ρ)-totally stable (in short, (K, ρ)-TS) if for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that if $t_0 \geq 0$, $\rho(x_{t_0}, u_{t_0}) < \delta(\epsilon)$ and $h \in BC([t_0, \infty))$ which satisfies $\sup_{t\in[t_0,\infty)}|h(t)| < \delta(\epsilon)$, then $\rho(x_t, u_t) < \epsilon$ for all $t \geq t_0$, where $x(t)$ is a solution of (6) through (t_0, ϕ) such that $x_{t_0}(s) = \phi(s) \in K$ for all $s \leq 0$.
If the above term $\rho(x_t, u_t)$ is replaced by $|x(t) - u(t)|$, then we have another concept of (K, ρ)-total stability; which will be referred to as the $((K, \rho), R^n)$-total stability (in short, $((K, \rho), R^n)$-TS).

Next, we shall consider the week stability concept than the total stability. For the compact set K, $(P, Q) \in \Omega(f)$, we define $\pi(P, Q)$ by

$$\pi(P, Q) = \sum_{j=1}^{\infty} \pi_j(P, Q)/[2^j(1 + \pi_j(P, Q))],$$

where $\pi_j(P, Q) = \sup\{|P(t, x_t(s)) - Q(t, x_t(s))| : t \in R, s \in [-j, 0], \text{ and } x_t(s) \in K\}$.

Definition 5. The bounded solution $u(t)$ of Eq.(5) is said to be BC-stable under disturbances from $\Omega(f)$ with respect to K (in short, BC-s.d.$\Omega(f)$) if for any $\epsilon > 0$ there exists an $\eta(\epsilon) > 0$ such that $|x(t) - u(t)| < \epsilon$ for all $t \geq t_0$, whenever $g \in \Omega(f)$, $\pi(f, g) \leq \eta(\epsilon)$ and $|x_{t_0} - u_{t_0}|_{BC} < \eta(\epsilon)$ for some $t_0 \geq 0$, where $x(t)$ is a solution through (t_0, ϕ) of

$$\dot{x}(t) = g(t, x_t), \qquad t \geq 0 \tag{7}$$

such that $x_{t_0}(s) = \phi(s) \in K$ for all $s \leq 0$.

Definition 6. The bounded solution $u(t)$ of Eq.(5) is said to be (K, ρ)-stable under disturbances from $\Omega(f)$ (in short, (K, ρ)-s.d.$\Omega(f)$) if for any $\epsilon > 0$ there exists an $\eta(\epsilon) > 0$ such that $\rho(x_t, u_t) < \epsilon$ for all $t \geq t_0$, whenever $g \in \Omega(f)$, $\pi(f, g) \leq \eta(\epsilon)$ and $\rho(x_{t_0}, u_{t_0}) < \eta(\epsilon)$ and for some $t_0 \geq 0$, where $x(t)$ is a solution of (7) through (t_0, ϕ) such that $x_{t_0}(s) = \phi(s) \in K$ for all $s \leq 0$.
If the above term $\rho(x_t, u_t)$ is replaced by $|x(t) - u(t)|$, then we have another concept of (K, ρ)-stable under disturbances from $\Omega(f)$; which will be referred to as the $((K, \rho), R^n)$-stable under disturbances from $\Omega(f)$ (in short, $((K, \rho), R^n)$-s.d.$\Omega(f)$).

Therefore the (K, ρ)-s.d.$\Omega(f)$ implies the BC-s.d.$\Omega(f)$, because of $\rho(\phi, \psi) \leq |\phi - \psi|_{BC}$ for $\phi, \psi \in BC$. In Theorem 3, we discuss the opposite implications.

Theorem 1. Under the assumptions (H1),(H2) and (H3), if the bounded solution $u(t)$ of Eq.(5) is (K, ρ)-TS, then it is (K, ρ)-s.d.$\Omega(f)$.

Proof. For a given $\epsilon > 0$, let $\delta(\epsilon)$ be the number for total stability of $u(t)$. For this $\delta(\epsilon) > 0$ and compact set K, it follows from (H1) that there exists an $S = S(\delta(\epsilon)/4, K) > 0$ such that, $-\infty \leq s \leq -S$,

$$|f(t, x_t(s))| \leq \delta(\epsilon)/4, \tag{8}$$

whenever $x(\sigma) \in K$ for all $\sigma \le t$. Also, for any $g \in \Omega(f)$ we have, $-\infty \le s \le -S$,

$$|g(t, x_t(s))| \le \delta(\epsilon)/4. \tag{9}$$

We can find the positive integer $N_0 = N_0(\epsilon)$ such that $[-S, 0] \subset [-N_0, 0]$. We set $\eta(\epsilon) = \min(\delta^*(\epsilon), \delta(\epsilon)/4)$, where $\delta^*(\epsilon) = (\delta(\epsilon)/4S)/2^N(1 + \delta(\epsilon)/4S)$. We shall show that if $g \in \Omega(f), \pi(f, g) \le \eta(\epsilon)$ and $\rho(u_\tau, y_\tau) \le \eta(\epsilon)$ for some $\tau \ge 0$, then $\rho(u_t, y_t) < \epsilon$ for all $t \ge \tau$, where $y(t)$ is a solution through (τ, y_τ) of

$$\dot{x}(t) = g(t, x_t)$$

such that $y_\tau(s) \in K$ for all $s \le 0$. On the other hand, $y(t)$ is a solution of

$$\dot{x}(t) = f(t, x_t) + h(t),$$

where

$$h(t) = g(t, y_t) - f(t, y_t).$$

Since $\pi(f, g) \le \eta(\epsilon)$, we have $\pi_N(f, g)/2^N(1 + \pi_N(f, g)) \le \delta^*(\epsilon)$. Thus $\pi_N(f, g) \le \delta(\epsilon)/4S$, that is, for $-N_0 \le s \le 0$,

$$\sup_{t \in R, x \in K} |f(t, x) - g(t, x)| \le \delta(\epsilon)/4S,$$

Thus, $-S \le s \le 0$,

$$|f(t, y_t(s)) - g(t, y_t(s))| \le \delta(\epsilon)/4S \tag{10}$$

as long as $y_t(s) \in K$. From (8),(9) and (10),

$$\begin{aligned} |f(t, y_t(s)) - g(t, y_t(s))| &\le |f(t, x_t(s))| + |g(t, x_t(s))| \\ &+ |f(t, y_t(s)) - g(t, y_t(s))| \le 3\delta(\epsilon)/4 \end{aligned} \tag{11}$$

as long as $y_t(s) \in K$. Thus, from (11), we have $|h(t)| \le \delta(\epsilon)$ for all $t \ge \tau$ as long as $y_t(s) \in K$. Since $u(t)$ is (K, ρ)-TS, $\rho(u_\tau, y_\tau) \le \delta(\epsilon)$ and $|h(t)| \le \delta(\epsilon)$, we obtain $\rho(u_t, y_t) \le \epsilon$ as long as $y_t(s) \in K$, which implies that $y(t)$ exists for all $t \ge \tau$ and $\rho(u_t, y_t) \le \epsilon$ for all $t \ge \tau$. This shows that $u(t)$ is (K, ρ)-s.d.$\Omega(f)$.

Theorem 2. Under the assumptions (H1),(H2) and (H3), if the bounded solution $u(t)$ of Eq.(5) is (K, ρ)-s.d.$\Omega(f)$, then it is an asymptotically almost periodic solution of Eq.(5). Consequently Eq.(5) has an almost periodic solution.

Proof. Let $\{t_k\}$ be any real sequence such that $t_k \to \infty$ as $k \to \infty$. If we set $u^k(t) = u(t + t_k)$, then $u^k(t)$ is a solution of

$$\dot{x}(t) = f(t + t_k, x_t) \tag{12}$$

through $(0, u_0^k)$ and $u_0^k(s) = u_{n_k}(s) \in K$ for all $s \leq 0$.
We claim that, under assumptions (H1),(H2) and (H3), suppose that the bounded solution $u(t)$ of (5) is (K, ρ)-s.d.$\Omega(f)$ and let a be a positive constant. Then $w(t) = u(t + a)$ is a solution of

$$\dot{x}(t) = f(t + a, x_t)$$

such that $w_0(s) = u_a(s) \in K$ for all $s \leq 0$, and it is (K, ρ)-s.d.$\Omega(f_a)$ for the same pair $(\epsilon, \eta(\epsilon))$ as the one for $u(t)$.
We shall show that if $r \in \Omega(f_a), \pi(f_a, r) \leq \eta(\epsilon)$ and $\rho(w_\tau, y_\tau) \leq \eta(\epsilon)$ for some $\tau \geq 0$, then $\rho(w_t, y_t) < \epsilon$ for all $t \geq \tau$, where $y(t)$ is a solution through (τ, y_τ) of

$$\dot{x}(t) = r(t, x_t)$$

such that $y_\tau(s) \in K$ for all $s \leq 0$.
If we set $z(t) = y(t - a)$, then $z(t)$ is defined on $t \geq \tau + a$ and is a solution through $(\tau + a, y_\tau)$ of

$$\dot{x}(t) = r(t - a, x_t)$$

such that $z_{\tau+a}(s) = y_\tau(s) \in K$ for all $s \leq 0$. On the other hand, if we set $g = r_{-a} \in \Omega(f)$, then $z(t)$ is a solution of

$$\dot{x}(t) = g(t, x_t)$$

such that $z_{\tau+a}(s) \in K$ for all $s \leq 0$. Since $\pi(f_a, r) \leq \eta(\epsilon), \pi(f, g) = \pi(f, r_{-a}) \leq \eta(\epsilon)$. Moreover, since $\rho(u_{\tau+a}, z_{\tau+a}) = \rho(w_\tau, y_\tau) \leq \eta(\epsilon)$ and $u(t)$ is (K, ρ)-s.d.$\Omega(f)$, we have $\rho(u_t, z_t) < \epsilon$ for all $t \geq \tau + a$, that is $\rho(w_t, y_t) < \epsilon$ for all $t \geq \tau$.
This show that $w(t)$ is (K, ρ)-s.d.$\Omega(f_a)$ for the same pair $(\epsilon, \eta(\epsilon))$.
By above claim, $u^k(t)$ is (K, ρ)-s.d.$\Omega(f_{t_k})$ for the same pair $(\epsilon, \eta(\epsilon))$ as the one for $u(t)$. Since $f(t, x)$ is almost periodic in t, there exists a subsequence of $\{t_k\}$, which we shall denote by $\{t_k\}$ again, such that $f(t + t_k, x)$ converges uniformly on $R \times K$, and hence for any $\epsilon > 0$ there exists a positive integer $k_1(\epsilon)$ such that if $k, m \geq k_1(\epsilon)$,

$$|f(t + t_k, x) - f(t + t_m, x)| < \eta(\epsilon)$$

for all $t \in R$ and $x \in K$. Thus we have $\pi(f_{t_k}, f_{t_m}) < \eta(\epsilon)$ if $k, m \geq k_1(\epsilon)$, since

$$\begin{aligned}
\pi(f_{t_k}, f_{t_m}) &\leq \sum_{j=1}^{N_1} \pi_j(f_{t_k}, f_{t_m})/2^j(1 + \pi_j(f_{t_k}, f_{t_m})) + \sum_{j=N_1+1}^{\infty} 1/2^j \\
&\leq \sum_{j=1}^{N_1} \pi_{N_1}(f_{t_k}, f_{t_m})/2^j + \eta(\epsilon)/2 < \eta(\epsilon),
\end{aligned}$$

where $N_1 = N_1(\epsilon)$ is a positive integer such that $\sum_{j=N_1+1}^{\infty} 1/2^j < \eta(\epsilon)/2$. Taking a subsequence of $\{t_k\}$ if necessary, we can assume that $u^k(s)$ converges uniformly on any compact interval in $(-\infty, 0]$. Therefore there exists a positive integer $k_2(\epsilon)$ such that if $k, m \geq k_2(\epsilon)$, we have $\rho(u_0^k, u_0^m) < \eta(\epsilon)$. On the other hand, $u^m(t) = u(t + t_m)$ is a solution of

$$\dot{x}(t) = f(t + t_m, x_t)$$

such that $u_0^m(s) \in K$ for all $s \leq 0$ and $(f_{t_m}) \in \Omega(f_{t_k}) = \Omega(f)$. Moreover, $\pi(f_{t_k}, f_{t_m}) < \eta(\epsilon)$ and $\rho(u_0^k, u_0^m) < \eta(\epsilon)$ if $k, m \geq k_0(\epsilon) = \max(k_1(\epsilon), k_2(\epsilon))$. Since u_t^k is (K, ρ)-s.d.$\Omega(f_{t_k})$, we have $\rho(u_t^k, u_t^m) < \epsilon$ for all $t \geq 0$ if $k, m \geq k_0(\epsilon)$. This implies that if $k, m \geq k_0(\epsilon)$,

$$|u(t + t_k) - u(t + t_m)| \leq \sup_{s \in [-1,0]} |u(t + s + t_k) - u(t + s + t_m)| < 4\epsilon$$

for all $\epsilon \leq 1/4$ and all $t \geq 0$. Thus we see that for any sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$, there exists a subsequence $\{t_{k_j}\}$ of $\{t_k\}$ for which $u(t + t_{k_j})$ converges uniformly on $[0, \infty)$ as $j \to \infty$. This shows that $u(t)$ is an asymptotically almost periodic solution of (5). Now we have

$$u(t) = p(t) + q(t),$$

where $p(t)$ is almost periodic in t and $q(t)$ is a function such that $q(t) \to 0$ as $t \to \infty$. There exists a sequence $\{t_k\}, t_k \to \infty$ as $k \to \infty$, such that $p(t + t_k) \to p(t)$ uniformly on $R, f(t + t_k, x) \to f(t, x)$ uniformly on $R \times S$ for any compact set S in B. Now we set $u^k(t) = u(t + t_k)$. Then $u^k(t)$ converges to $p(t)$ uniformly on any compact set in R as $k \to \infty$, and $u^k(t)$ is a solution of (12). Thus we can show that $p(t)$ is a solution of (5). This shows that the equation (5) has an almost periodic solution.

Corollary 1. Under the assumptions (H1),(H2) and (H3), if the bounded solution $u(t)$ of Eq.(5) is (K, ρ)-TS, then it is an asymptotically almost periodic solution of Eq.(5) and Eq.(5) has an almost periodic solution.

Proof. It is easy from Theorem 1 and 2 to prove this corollary. Indeed, the (K, ρ)-TS of $u(t)$ yields the (K, ρ)-s.d.$\Omega(f)$ of $u(t)$ by Theorem 1, and then $u(t)$ is an asymptotic almost periodic solution of Eq.(5) by Theorem 2. Therefore Eq.(5) has an almost periodic solution.

Theorem 3. Let B be a fading memory space, and assume conditions (H1),(H2) and (H3). Then the solution $u(t)$ of Eq.(5) is BC-s.d.$\Omega(f)$ implies the solution $u(t)$ of Eq.(5) is (K, ρ)-s.d.$\Omega(f)$.

Proof. First, we will prove the following claim.

Claim 1. Under the above assumption, the solution $u(t)$ of Eq.(5) is $((K,\rho), R^n)$-s.d.$\Omega(f)$ implies (K,ρ)-s.d.$\Omega(f)$.

Take any $\epsilon > 0, (\tau, \phi) \in R^+ \times BC$ and $g \in \Omega(f)$ with $\phi(s) = x_\tau(s) \in K$ for all $s \leq 0$, $\rho(\phi, u_\tau) < \delta(\epsilon)$ and $\pi(f,g) < \delta(\epsilon)$, where $\delta(\cdot)$ is the one for $((K,\rho), R^n)$-s.d.$\Omega(f)$ of the solution $u(t)$ of (5). Then $x(t)$ of solution through (τ, ϕ) of (7) satisfies

$$|x(t) - u(t)| < \epsilon \qquad \text{for} \quad t \geq \tau. \tag{13}$$

To estimate $\rho(x_t, u_t)$, we first estimate $|x_t - u_t|_j$. Let $t \geq \tau$, and denote by k is the largest integer which does not exceed $t - \tau$. If $j \leq k$, then $j \leq t - \tau$; hence $|x_t - u_t|_j = \sup_{-j \leq s \leq 0} |x(t+s) - u(t+s)| < \epsilon$ by (13). On the one hand, if $j \geq k+1$, then $j > t - \tau$; hence

$$\begin{aligned} |x_t - u_t|_j &= \max\{\sup_{-j \leq s \leq \tau - t} |x(t+s) - u(t+s)|, \sup_{\tau - t \leq s \leq 0} |x(t+s) - u(t+s)|\} \\ &\leq \max\{\sup_{-j \leq \theta \leq 0} |\phi(\theta) - u(\tau+\theta)|, \sup_{\tau \leq \theta} |x(\theta) - u(\theta)|\} < |\phi - u_\tau|_j + \epsilon \end{aligned}$$

by (13). Then

$$\begin{aligned} \rho(x_t, u_t) &= (\sum_{j=1}^{k} + \sum_{j=k+1}^{\infty}) 2^{-j} |x_t - u_t|_j / [1 + |x_t - u_t|_j] \\ &< \sum_{j=1}^{k} 2^{-j} \epsilon/(1+\epsilon) + \sum_{j=k+1}^{\infty} 2^{-j} [|\phi - u_\tau|_j + \epsilon] / [1 + |\phi - u_\tau|_j + \epsilon] \\ &\leq \sum_{j=1}^{\infty} 2^{-j} \epsilon/(1+\epsilon) + \sum_{j=k+1}^{\infty} 2^{-j} |\phi - u_\tau|_j / [1 + |\phi - u_\tau|_j < \epsilon + \delta(\epsilon) \leq 2\epsilon, \end{aligned}$$

which shows that the solution $u(t)$ of Eq.(5) is (K,ρ)-s.d.$\Omega(f)$ with $\delta(\cdot/2)$.

Now, in order to complete the proof of Theorem 3, we shall accomplish it by contradiction. By claim 1, we assume that the solution $u(t)$ of Eq.(5) is BC-s.d.$\Omega(f)$ but not $((K,\rho), R^n)$-s.d.$\Omega(f)$ here, $K \subset \{x \in R^n : |x| \leq \alpha\}$ for some $\alpha > 0$. Since the solution $u(t)$ of Eq.(5) is not $((K,\rho), R^n)$-s.d.$\Omega(f)$, there exists an $\epsilon \in (0,1)$, sequence $\{\tau_m\} \subset R^+, \{t_m\}(t_m > \tau_m), \{\phi^m\} \subset BC$ with $\phi^m(s) = x_{\tau_m}(s) \in K$ for all $s \leq 0$, $\{g_m\}$ with $g_m \in \Omega(f)$, and solutions $\{x(t)\}$ through (τ_m, ϕ^m) of

$$\dot{x}(t) = g_m(t, x_t) \tag{14}$$

such that

$$\rho(\phi^m, u_{t_m}) < 1/m \quad \text{and} \quad \pi(f_m, g_m) < 1/m \tag{15}$$

and that

$$|x(t_m) - u(t_m)| = \epsilon \quad \text{and} \quad |x(t) - u(t)| < \epsilon \quad \text{on} \quad [\tau_m, t_m) \quad (16)$$

for $m \in N$ (N denotes the set of all positive integers), where $x(t_m)$ is a solution through (τ_m, ϕ^m) of (14). For each $m \in N$ and $w \in R^+$, we define $\phi^{m,w} \in BC$ by

$$\phi^{m,w}(\theta) = \begin{cases} \phi^m(\theta) & \text{if} \quad -w \le \theta \le 0, \\ \phi^m(-w) + u(\tau_m + \theta) - u(\tau_m - w) & \text{if} \quad \theta < -w. \end{cases}$$

Notice that $|\phi^{m,w} - u_{\tau_m}|_{BC} = |\phi^m - u_{\tau_m}|_{[-w,0]}$.

Claim 2. $\sup\{|\phi^{m,w} - \phi^m|_B : m \in N\} \to 0$ as $w \to \infty$.

If this is not true, there exist an $\epsilon > 0$ and sequences $\{m_k\} \subset N$ and $\{w_k\}$, $w_k \to \infty$ as $k \to \infty$, such that $|\phi^{m_k,w_k} - \phi^{m_k}|_B \ge \epsilon$ for $k = 1, 2, \cdots$. Put $\psi^k = \phi^{m_k,w_k} - \phi^{m_k}$. Clearly, $\{\psi^k\}$ is a sequence in BC which converges to the zero function compactly on R^- and $\sup_k |\psi^k|_{BC} < \infty$. Then axiom (A2) yields that $|\psi^k|_B \to 0$ as $k \to \infty$, a contradiction.

Claim 3. The set $\{\phi^{m,w}, \phi^m : m \in N, w \in R^+\}$ is relatively compact in B.

Indeed, since the set $\Gamma(u)$ is compact in B, (15) and axiom (A2) yield that any sequence $\{\phi^{m_j}\}_{j=1}^{\infty} (m_j \in N)$ has a convergent subsequence in B. Therefore, it suffices to show that any sequence $\{\phi^{m_j,w_j}\}_{j=1}^{\infty} (m_j \in N, w_j \in R^+)$ has a convergent subsequence in B. We assert that the sequence of functions $\{\phi^{m_j,w_j}(\theta)\}_{j=1}^{\infty}$ contains a subsequence which is equicontinuous on any compact set in R^-. If this is the case, then the sequence $\{\phi^{m_j,w_j}\}_{j=1}^{\infty}$ would have a convergent subsequence in B by Ascoli's theorem and axiom (A2), as required. Now, notice that the sequence of functions $\{u(\tau_{m_j} + \theta)\}$ is equicontinuous on any compact set in R^-. Then the assertion obviously holds true when the sequence $\{m_j\}$ is bounded. Taking a subsequence if necessary, it is thus sufficent to consider the case $m_j \to \infty$ as $j \to \infty$. In this case, from (15) it follows that $\phi^{m_j}(\theta) - u(\tau_{m_j} + \theta) =: v^j(\theta) \to 0$ uniformly on any compact set in R^-. Consequently, $\{v^j(\theta)\}$ is equicontinuous on any compact set in R^-, and so is $\{\phi^{m_j}(\theta)\}$. Therefore the assertion immediately follows from this observation.

Now, for any $m \in N$, set the solution $x^m(t) = x^m(t + \tau_m)$ of (14) if $t \le t_m - \tau_m$ and $x^m(t) = x^m(t_m - \tau_m)$ if $t > t_m - \tau_m$. Moreover, set $x^{m,w}(t) = \phi^{m,w}(t)$ if $t \in R^-$ and $x^{m,w}(t) = x^m(t)$ if $t \in R^+$. Since $x_0^m = \phi^m$ and $|x^m(t)| < 1 + |u|_{[0,\infty)} =: d < \infty$ for $t \in R^+$, we have

$$|x_t^m|_B \le Ld + M|\phi^m|_B \le Ld + Ml|\phi^m|_{BC} \le Ld + Ml\alpha$$

by (1) and axiom (A1) hence, if $0 \leq t < t_m - \tau_m$, then

$$\begin{aligned} |(d/dt)x^m(t)| &\leq |f(t+\tau_m, x_t^m)| + |g_m(t+\tau_m, x_t^m) - f(t+\tau_m, x_t^m)| \\ &\leq L_0(Ld + Ml\alpha) + 1/m \leq L_1 \quad \text{(independent of } m \in N) \end{aligned}$$

by (15) and (H1). Consequently,

$$|x^m(s_1) - x^m(s_2)| \leq L_1|s_1 - s_2|, \qquad s_1, s_2 \in R^+, m \in N. \quad (17)$$

Set

$$W = \text{the closure of } \{x_t^{m,w}, x_t^m : m \in N, t \in R^+, w \in R^+\}.$$

Combining (17) with Claim 3, we see by (cf.[5,6,7,8]) that the set K is compact in B, hence $f(t, \phi)$ is uniformly continuous on $R^+ \times K$ by (H2). Define a function $q_{m,w}$ on R^+ by $q_{m,w}(t) = f(t+\tau_m, x_t^m) - f(t+\tau_m, x_t^{m,w})$ if $0 \leq t \leq t_m - \tau_m$, and $q_{m,w}(t) = q_{m,w}(t_m - \tau_m)$ if $t > t_m - \tau_m$. Since $|x_t^{m,w} - x_t^m|_B \leq M|\phi^{m,w} - \phi^m|_B$ $(t \in R^+, m \in N)$ by axiom (A1), if follows from Claim 2 that $\sup\{|x_t^{m,w} - x_t^m|_B : t \in R^+, m \in N\} \to 0$ as $w \to \infty$; hence one can choose $w = w(\epsilon) \in N$ in such a way that

$$\sup\{|\hat{q}_{m,w}(t)| : m \in N, t \in R^+\} < \delta(\epsilon/2)/2,$$

where $\delta(\cdot)$ is the one for BC-s.d.$\Omega(f)$ of the solution $u(t)$ of Eq.(5). Moreover, for this w, select an $m \in N$ such that $m > 2^w(1+\delta(\epsilon/2))/\delta(\epsilon/2)$. Then $2^{-w}|\phi^m - u_{\tau_m}|_w/[1 + |\phi^m - u_{\tau_m}|_w] \leq \rho(\phi^m, u_{\tau_m}) < 2^{-w}\delta(\epsilon/2)/[1+\delta(\epsilon/2)]$ by (15), which implies that

$$|\phi^m - u_{\tau_m}|_w < \delta(\epsilon/2) \quad \text{or} \quad |\phi^{m,w} - u_{\tau_m}|_{BC} < \delta(\epsilon/2).$$

The function $x^{m,w}$ satisfies $x_0^{m,w} = \phi^{m,w}$ and

$$\begin{aligned} (d/dt)x^{m,w}(t) &= (d/dt)x^m(t) = f_m(s+\tau_m, x_s^m) + (g_m(s+\tau_m, x_s^m) - f_m(s+\tau_m, x_s^m)) \\ &= f_m(s+\tau_m, x_s^{m,w}) + q_{m,w}(s) + (g_m(s+\tau_m, x_s^m) - f_m(s+\tau_m, x_s^m)) \end{aligned}$$

for $t \in [0, t_m - \tau_m)$. Since $u^m(t) = u(t+\tau_m)$ is a BC-s.d.$\Omega(f)$ solution of

$$\dot{x}(t) = f_m(t+\tau_m, x_t^m)$$

with the same $\delta(\cdot)$ as the one for $u(t)$, from the fact that $\pi(f_m, g_m) < 1/m$ and hence $\sup_{t\geq 0}|q_{m,w}(t) + (g_m(t+\tau_m, x_t^m) - f_m(t+\tau_m, x_t^m))| < \delta(\epsilon/2)/2 + 1/m < \delta(\epsilon/2)$ it follows that $|x^{m,w}(t) - u(t+\tau_m)| < \epsilon/2$

on $[0, t_m - \tau_m)$. In particular, we have $|x^{m,w}(t_m - \tau_m) - u(t_m)| < \epsilon$ or $|x(t_m) - u(t_m)| < \epsilon$, which contradicts (16). This completes the proof.

This theorem is true for functional difference equations with infinite delay on BS [4] and also for abstract functional differential equations with infinite delay [3]. By Theorem 3 and Theorem 2, we have the following corollary.

Corollary 2. Let B be a fading memory space. Under the assumptions (H1),(H2) and (H3), if the bounded solution $u(t)$ of equation (5) is BC-s.d.$\Omega(f)$, then the Eq.(5) has an almost periodic solution.

[9] has established that if the bounded solution $u(t)$ of $\dot{x}(t) = f(t, x_t)$ is BC-TS then it is (K, ρ)-TS, and it also is well known that if the bounded solution $u(t)$ of the above equation is B-TS, then it is B-s.d.$\Omega(f)$ [6]. We can improve these to equation (5). Then, we have the following corollary.

Corollary 3. Let B be a fading memory space. Under the assumptions (H1),(H2) and (H3), if the bounded solution $u(t)$ of Eq.(5) is BC-TS, then the Eq.(5) has an almost periodic solution.

Proof. The BC-TS of $u(t)$ implies the BC-s.d.$\Omega(f)$ of $u(t)$ by (cf.[Corollary 1,10 in 6] and Theorem 1), then $u(t)$ is an asymptotically almost periodic solution of Eq.(5) by Corollary 2. Therefore Eq.(5) has an almost periodic solution.

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